First Semester Examination
Academic Session 2018/2019

December 2018/January 2019

MAT201 - Advanced Calculus
(Kalkulus Lanjutan)

Duration : 3 hours
(Masa : 3 jam)

Please check that this examination paper consists of SEVEN (7) pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH (7) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **ALL SIX (6)** questions.

*[**Arahan:** Jawab **SEMUA ENAM (6)** soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

Question 1

(a) (i) Show that $\lim_{t\to 0^+}(t^3 \ln t) = 0$.

(ii) Find the indefinite integral $\int x^2 \ln x \, dx$.

(iii) Determine whether the improper integral $\int_0^2 x^2 \ln x \, dx$ is convergent or divergent.

[40 marks]

(b) By using the Comparison Theorem, determine whether the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent or divergent.

[20 marks]

(c) A sequence $\langle a_n \rangle_{n=1}^\infty$ is recursively defined by

$$a_1 = \sqrt{2} \text{ and } a_{n+1} = \sqrt{2 + a_n} \text{ for all } n \geq 1.$$

(i) Show that $a_n < 2$ for all $n \geq 1$.

(ii) Given the fact that this sequence is increasing, deduce that $\langle a_n \rangle_{n=1}^\infty$ is convergent and find its limit.

[40 marks]

Soalan 1

(a) (i) Tunjukkan bahawa $\lim_{t\to 0^+}(t^3 \ln t) = 0$.

(ii) Dapatkan kamiran tak tentu $\int x^2 \ln x \, dx$.

(iii) Tentukan sama ada kamiran tak wajar $\int_0^2 x^2 \ln x \, dx$ *adalah menumpu atau mencapah.*

[40 markah]

(b) Dengan menggunakan Teorem Perbandingan, tentukan sama ada kamiran tak wajar $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ *adalah menumpu atau mencapah.*

[20 markah]

(c) *Suatu jujukan* $\langle a_n \rangle_{n=1}^{\infty}$ *ditakrifkan secara rekursi oleh*

$$a_1 = \sqrt{2} \text{ dan } a_{n+1} = \sqrt{2 + a_n} \text{ untuk semua } n \geq 1.$$

(i) *Tunjukkan bahawa* $a_n < 2$ *untuk semua* $n \geq 1$.

(ii) *Diberikan fakta bahawa jujukan ini adalah menokok, deduksikan bahawa* $\langle a_n \rangle_{n=1}^{\infty}$ *adalah menumpu dan dapatkan had jujukan tersebut.*

[40 markah]

Question 2

(a) Evaluate the series $\sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$.

[30 marks]

(b) Determine whether the given series is convergent or divergent.

(i) $\sum_{n=1}^{\infty} \frac{5 + 2n^3}{n^5 + 3n^2 + 7}$

(ii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

[60 marks]

(c) Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series, where $a_n \neq 0$ for all $n \geq 1$. Prove that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a divergent series.

[10 marks]

Soalan 2

(a) *Nilaikan siri* $\sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$.

[30 markah]

(b) *Tentukan sama ada siri yang diberi menumpu atau mencapah.*

(i) $\sum_{n=1}^{\infty} \frac{5 + 2n^3}{n^5 + 3n^2 + 7}$

(ii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

[60 markah]

(c) *Andaikan bahawa* $\sum_{n=1}^{\infty} a_n$ *ialah suatu siri menumpu dengan* $a_n \neq 0$ *untuk semua* $n \geq 1$. *Buktikan bahawa* $\sum_{n=1}^{\infty} \frac{1}{a_n}$ *ialah suatu siri mencapah.*

[10 markah]

Question 3

(a) (i) Justify that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.

(ii) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}$.

[40 marks]

(b) (i) Derive the Maclaurin series for $f(x) = \ln(1+x)$ using the definition. Then find its radius of convergence.

(ii) Using your answer in part (i), find the Maclaurin series for $f(x) = x\ln(1-3x)$ and its radius of convergence.

[45 marks]

(c) Give an example of a conditionally convergent series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} na_n$ converges.

[15 marks]

Soalan 3

(a) *(i)* *Justifikasikan bahawa siri* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *adalah menumpu.*

(ii) *Cari selang penumpuan untuk siri kuasa* $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}$.

[40 markah]

(b) *(i)* *Terbitkan siri Maclaurin untuk* $f(x) = \ln(1+x)$ *dengan menggunakan takrif. Kemudian, dapatkan jejari penumpuan siri ini.*

(ii) *Dengan menggunakan jawapan anda di bahagian (i), cari siri Maclaurin untuk* $f(x) = x\ln(1-3x)$ *dan jejari penumpuan siri ini.*

[45 markah]

(c) *Berikan satu contoh siri menumpu bersyarat* $\sum_{n=1}^{\infty} a_n$ *sedemikian hingga* $\sum_{n=1}^{\infty} na_n$ *menumpu.*

[15 markah]

Question 4

(a) Determine the region where the following function is continuous.

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+xy+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

[40 marks]

(b) Consider the function $f(x,y)=x^4y^2-x^3\cos(y)$. Find

(i) f_{xxx},

(ii) f_{xyx}.

[30 marks]

(c) If $z=f(x,y)$, where $x=s+t$ and $y=s-t$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\frac{\partial z}{\partial s}\frac{\partial z}{\partial t}.$$

[30 marks]

Soalan 4

(a) Tentukan rantau supaya fungsi berikut selanjar.

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+xy+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

[40 markah]

(b) Pertimbangkan fungsi $f(x,y)=x^4y^2-x^3\cos(y)$. *Dapatkan*

(i) f_{xxx},

(ii) f_{xyx}.

[30 markah]

(c) Jika $z=f(x,y)$ *dengan* $x=s+t$ *dan* $y=s-t$, *tunjukkan bahawa*

$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\frac{\partial z}{\partial s}\frac{\partial z}{\partial t}.$$

[30 markah]

Question 5

(a) Find the directional derivative of $f(x,y) = \sqrt{xy}$ at $P(2,8)$ in the direction from P to $Q(5,4)$.

[30 marks]

(b) Let $f(x,y) = x + 4y + \dfrac{2}{xy}$.

(i) Find the critical point(s) of $f(x,y)$.

(ii) Use the second derivative test to classify the type of critical point(s) found in part (a).

[40 marks]

(c) Find the point on the plane $2x + y - z = 6$ which is the closest to the origin, by using the method of Lagrange multipliers.

[30 marks]

Soalan 5

(a) Dapatkan terbitan berarah bagi $f(x,y) = \sqrt{xy}$ pada $P(2,8)$ dalam arah dari P ke $Q(5,4)$.

[30 markah]

(b) Biar $f(x,y) = x + 4y + \dfrac{2}{xy}$.

(i) Dapatkan titik genting bagi $f(x,y)$.

(ii) Gunakan ujian terbitan kedua untuk mengkelaskan jenis titik genting yang ditemui di bahagian (a).

[40 markah]

(c) Dapatkan titik terdekat kepada titik asalan pada satah $2x + y - z = 6$ dengan menggunakan kaedah pendaraban Lagrange.

[30 markah]

Question 6

(a) Evaluate the following integral.

$$\int_0^{\pi/2} \int_0^x x \sin(y)\, dy\, dx.$$

[30 marks]

(b) Evaluate the following integral by changing the order of integration.

$$\int_0^3 \int_{x^2}^9 xe^{-y^2}\, dy\, dx.$$

[35 marks]

(c) Use polar coordinate to find the volume of solid below the paraboloid $z = 25 - 5x^2 - 5y^2$ and above the xy-plane.

[35 marks]

Soalan 6

(a) Nilaikan kamiran berikut.

$$\int_0^{\pi/2} \int_0^x x \sin(y)\, dy\, dx.$$

[30 markah]

(b) Nilaikan kamiran berikut dengan menukarkan turutan kamiran.

$$\int_0^3 \int_{x^2}^9 xe^{-y^2}\, dy\, dx.$$

[35 markah]

(c) Gunakan koordinat kutub untuk mencari isipadu bagi pepejal di bawah paraboloid $z = 25 - 5x^2 - 5y^2$ dan di atas satah xy.

[35 markah]

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